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HARMONIC FUNCTIONS FOR GENERALISED MEHLER SEMIGROUPS

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We study harmonic functions for generalised Mehler semigroups in infinite dimensions. The class of generalised Mehler semigroups includes transition semigroups determined by infinite dimensional Ornstein-Uhlenbeck processes perturbed by a Lévy noise. We prove results about existence and nonexistence of nonconstant bounded harmonic functions and establish convexity of positive harmonic functions. The paper extends some results proved by E. Priola and J. Zabczyk (J. Funct. Anal. 216 (2004)) to a separable Hilbert space setting.

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Harmonic functions for generalised Mehler semigroups

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Abstract: We study harmonic functions for generalised Mehler semigroups in infinite dimensions. The class of generalised Mehler semigroups includes transition semigroups determined by infinite dimensional Ornstein-Uhlenbeck processes perturbed by a Lévy noise. We prove results about existence and nonexistence of nonconstant bounded harmonic functions and establish convexity of positive harmonic functions. The paper extends some results proved in [27] to a separable Hilbert space setting.

1 Introduction

The classical Liouville theorem for the Laplace operator L states that if, for a bounded C^2-function u,

$$Lu(x) = 0, \;\; x \in \mathbb{R}^n,$$

then u is constant on $\mathbb{R}^n$. This result can be equivalently formulated in terms of the heat semigroup P_t,

$$P_t u(x) = \frac{1}{\sqrt{(2\pi t)^n}} \int_{\mathbb{R}^n} u(y) e^{\frac{|x-y|^2}{2t}} dy, \;\; t > 0, \quad P_0 u(x) = u(x), \; x \in \mathbb{R}^n, \; t \geq 0,$$

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i.e., if, for a bounded Borel function u, one has $P_t u(x) = u(x)$, for all $t \geq 0$, $x \in \mathbb{R}^n$, then u is constant on $\mathbb{R}^n$.

More generaly, let E be a Polish space and let P_t be a Markov semigroup, acting on the space $\mathcal{B}_b(E)$ of all real Borel and bounded functions defined on E. A bounded from below function $u : E \to \mathbb{R}$ is said to be *harmonic* for P_t, if u is Borel and invariant for P_t, i.e.,

$$P_t u(x) = u(x), \quad t \geq 0, \; x \in E. \tag{1.1}$$

We say that a harmonic function u is a *bounded harmonic function* (BHF) or a *positive harmonic function* (PHF) for P_t if in addition u is bounded or nonnegative. Note that if u is a BHF for P_t, then

$$Lu(x) = 0, \;\; x \in E,$$

where the operator L is defined as follows:

$$Lu(x) = \lim_{t \to 0^+} \frac{P_t u(x) - u(x)}{t}, \;\; x \in E. \tag{1.2}$$

A converse statement is true as well, see Section 3. Preliminaries are gathered in Section 2.

Our main concern in the present paper are harmonic functions for generalized Mehler semigroups introduced in [5]. They have recently received a lot of attention, see for instance [32], [9], [17], [21], [28] and references therein. This class includes transition semigroups determined by infinite dimensional Ornstein-Uhlenbeck processes perturbed by a Lévy noise. Those processes are solutions to the following infinite dimensional stochastic differential equation on a Hilbert space H,

$$dX_t = AX_t dt + BdW_t + CdZ_t, \;\; X_0 = x \in H, \; t \geq 0. \tag{1.3}$$

Here A generates a $\mathcal{C}_0$-semigroup e^{tA} on H, B and C are bounded linear operators from another Hilbert space U into H. Moreover W_t and Z_t are independent processes; W_t is a U-valued Wiener process and Z_t is a U-valued Lévy process (without a Gaussian component).

One says that the transition semigroup P_t has the Liouville property if all BHFs for P_t are constant. The Liouville property has been studied for various classes of linear and nonlinear operators L on $\mathbb{R}^n$. In particular, second order elliptic operators on $\mathbb{R}^n$, or on differentiable manifolds E, have been intensively investigated, see for instance [23], [6], [1], [31], [3], [18] and references therein. Liouville theorems for nonlocal operators are given in [2] and [27]. The probabilistic interpretation of the Liouville property is discussed in [27], see also [23]. A Liouville theorem for the infinite dimensional heat semigroup has already been considered in [12]. For connections between the Liouville property and the existence of invariant ergodic measures, see also Remark 4.4.

Theorem 4.1 of Section 4 is our main result on the Liouville property. In the particular case of an Ornstein-Uhlenbeck process X_t perturbed by a Lévy noise, see (1.3), and under suitable assumptions, the theorem states that the corresponding transition semigroup P_t has the Liouville property *if and only if* all λ in the spectrum $\sigma(A)$ of A have nonpositive real part. Moreover, when there exists $\lambda \in \sigma(A)$ with positive real part, we are able to construct a nonconstant BHF for P_t. This theorem extends to infinite dimensions a result given in [27].

In Section 5, we prove a result concerning positive harmonic functions. Under the assumptions of Theorem 4.1, we show that all PHFs for the transition semigroup P_t associated to (1.3) are convex. This result can be regarded as a stronger version of the first part of Theorem 4.1, see also Corollary 5.3.

The final section contains two open questions.

2 Preliminaries

Let H be a real separable Hilbert with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. We will identify H with H^* (the topological dual of H). Let U be another separable Hilbert space. By $\mathcal{L}(U,H)$ we denote the space of all bounded linear operators from U into H. We set $\mathcal{L}(H,H)=\mathcal{L}(H)$. If $B\in\mathcal{L}(U,H)$, its adjoint operator is denoted by B^* ($B^*\in\mathcal{L}(H,U)$).

The space $\mathcal{C}_b(H)$ (resp. $\mathcal{B}_b(H)$) stands for the Banach space of all real, continuous (resp. Borel) and bounded functions $f:H\to\mathbb{R}$, endowed with the supremum norm: $\|f\|_0=\sup_{x\in H}|f(x)|$.

The space $\mathcal{C}_b^k(H)$ is the set of all k-times differentiable functions f, whose Fréchet derivatives $D^i f$, $1\le i\le k$, are continuous and bounded on H, up to the order $k\ge 1$. Moreover we set $\mathcal{C}_b^\infty(H)=\cap_{k\ge 1}\,\mathcal{C}_b^k(H)$.

2.1 Characteristic functions

We collect some basic facts about characteristic functions in infinite dimensions. These will be used in the sequel, see [22] or [7] for more details.

A function $\psi:H\to\mathbb{C}$ is said to be *negative definite* if, for any $h_1,\dots h_n\in H$, c_1, $\dots c_n\in\mathbb{C}$, verifying $\sum_{k=1}^n c_k=0$, one has: $\sum_{i,j=1}^n\psi(h_i-h_j)c_i\overline{c_j}\le 0$.

A function $\theta:H\to\mathbb{C}$ is said to be *positive definite* if, for any $h_1,\dots h_n\in H$, the $n\times n$ Hermitian matrix $(\theta(h_i-h_j))_{ij}$ is positive definite. Remark that $\psi:H\to\mathbb{C}$ is negative definite if and only if the function $\exp(-t\psi(\cdot))$ is positive definite for any $t>0$.

A mapping $g:H\to\mathbb{C}$ is said to be *Sazonov continuous* on H if it is continuous with respect to the locally convex topology on H generated by the seminorms $p(x)=|Sx|$, $x\in H$, where S ranges over the family of all Hilbert-Schmidt operators on H. Of course any Sazonov continuous function is in particular continuous.

The Bochner theorem states that any function $f:H\to\mathbb{C}$ is the *characteristic function* of a probability measure μ on H, i.e.,

$$\hat\mu(h)=\int_H e^{i\langle y,h\rangle}\mu(dy)=f(h),\quad h\in H,$$

if and only if f is positive definite, Sazonov continuous and such that $f(0)=1$.

Let Q be a symmetric nonnegative definite trace class operator on H, we denote by $N(x,Q)$, $x\in H$, the *Gaussian measure* on H with mean x and covariance operator Q. The trace of Q will be denoted by $\mathrm{Tr}\,(Q)$.

2.2 Mehler semigroups

A Lévy process Z_t with values in H is a H-valued process defined on some stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$, continuous in probability, having stationary independent increments, càdlàg trajectories, and such that $Z_0=0$.

One has that

$$\mathbb{E}e^{i\langle Z_t,s\rangle} = \exp(-t\psi(s)),\ s \in H, \tag{2.1}$$

where $\psi : H \to \mathbb{C}$ is a Sazonov continuous, negative definite function such that $\psi(0) = 0$. We call ψ the *exponent* of Z_t. Viceversa given ψ with the previous properties, there exists a unique in law H-valued Lévy process Z_t, such that (2.1) holds.

The exponent ψ can be expressed by the following infinite dimensional Lévy-Khintchine formula,

$$\psi(s) = \frac{1}{2}\langle Qs, s\rangle - i\langle a, s\rangle - \int_H \Big(e^{i\langle s,y\rangle} - 1 - \frac{i\langle s, y\rangle}{1 + |y|^2}\Big)M(dy),\ \ s \in H, \tag{2.2}$$

where Q is a symmetric nonnegative definite trace class operator on H, $a \in H$ and M is the spectral Lévy measure on H associated to Z_t, see also [29].

A *generalised Mehler semigroup* S_t, acting on $\mathcal{B}_b(H)$, is given by

$$S_t f(x) = \int_H f(e^{tA}x + y)\mu_t(dy),\ \ t \geq 0,\ x \in H,\ f \in \mathcal{B}_b(H), \tag{2.3}$$

where e^{tA} is a $\mathcal{C}_0$-semigroup on H, with generator A, μ_t, $t \geq 0$, is a family of probability measures on H, such that

$$\hat{\mu}_t(h) = \exp\Big(-\int_0^t \psi(e^{sA^*}h)ds\Big),\ \ h \in H,\ t \geq 0. \tag{2.4}$$

Here $\psi : H \to \mathbb{C}$ is a continuous, negative definite function such that $\psi(0) = 0$. We call ψ the *exponent* of S_t. Note that we are not assuming that the exponent ψ is Sazonov continuous, i.e., we are not requiring that $\exp(-\psi(\cdot))$ is the characteristic function of a probability measure on H or, equivalently, that there exists an associated H-valued Lévy process.

Generalized Mehler semigroups were introduced in [5], see also [32], [9], [17], [21] and [28].

3 Abstract Liouville theorems

Here, combining arguments from [14] and [24], we prove an abstract result which allows to formulate the Liouville problem in terms of generators, see in particular Theorem 3.1. We also provide an application to an infinite dimensional Ornstein-Uhlenbeck operator.

Let P_t be any Markov semigroup acting on $B_b(E)$, the space of all bounded Borel functions on a Polish space E. Define the subspace

$$\mathcal{B}^0(E) = \{f \in \mathcal{B}_b(E),\quad \text{such that, for any } x \in E, \text{ the map: } t \mapsto P_t f(x) \text{ is continuous on } [0,\infty)\}. \tag{3.1}$$

This space is a slight modification of the space $\mathcal{B}_b^0(E)$ introduced in [14], see also Remark 3.3. It is easy to verify that the space $\mathcal{B}^0(E)$ is invariant for P_t. Moreover it is a closed subspace of $\mathcal{B}_b(E)$ with respect to the supremum norm. This space also satisfies the assumptions (i) and (ii) in [24, Section 5].

We consider P_t acting on $\mathcal{B}^0(E)$ and define a generator $L : D(L) \subset \mathcal{B}^0(E) \to \mathcal{B}^0(E)$ of P_t as a version of the Dynkin weak generator, by the formula:

$$\begin{aligned} D(L) &:= \Big\{u \in \mathcal{B}^0(E) : \sup_{t>0} \Big\| \frac{P_t u - u}{t} \Big\|_0 < \infty, \ \exists\, g \in \mathcal{B}^0(E) \text{ such that} \\ &\qquad \lim_{t \to 0^+} \frac{P_t u(x) - u(x)}{t} = g(x), \ \forall\, x \in E\Big\}, \\ Lu(x) &= \lim_{t \to 0^+} \frac{P_t u(x) - u(x)}{t}, \quad \text{for } u \in D(L), \ \ x \in E. \end{aligned} \tag{3.2}$$

We have the following characterisation.

Theorem 3.1 *If $f \in \mathcal{B}_b(E)$ then*

$$f \in D(L) \text{ and } \ Lf = 0 \quad \Longleftrightarrow \quad f \ \textit{ is a BHF for } P_t.$$

The theorem is a direct corollary of the following proposition.

Proposition 3.2 *For any function $f \in B_b(E)$, the following statements are equivalent:*

(i) $f \in D(L)$;
(ii) there exists $g \in \mathcal{B}^0(E)$ such that

$$P_t f(x) - f(x) = \int_0^t P_s g(x) ds, \quad x \in E, \ t \geq 0. \tag{3.3}$$

Moreover if (3.3) holds then $Lf = g$.

Proof $(ii) \Rightarrow (i)$. By (3.3) one has that $f \in \mathcal{B}^0(E)$. Moreover $\frac{P_t f(x) - f(x)}{t} \to g(x)$, as $t \to 0^+$, for any $x \in E$. Finally, there results:

$$\sup_{t>0} \Big\| \frac{P_t f - f}{t} \Big\|_0 \leq \sup_{s>0} \|P_s g\|_0 \leq \|g\|_0.$$

$(i) \Rightarrow (ii)$. Fix $x \in E$. Note that

$$\lim_{t \to 0^+} P_s \Big(\frac{P_t f - f}{t} \Big)(x) = P_s L f(x), \quad s \geq 0.$$

Hence, there exists the right derivative $\partial_s^+ P_s f(x) = P_s L f(x)$, $s \geq 0$. Since the functions: $s \mapsto P_s f(x)$ and $s \mapsto P_s L f(x)$ are both continuous on $[0, +\infty)$, by a well known lemma of Real Analysis, the function: $s \mapsto P_s f(x)$ is $\mathcal{C}^1([0, +\infty))$. This gives the assertion. ■

Remark 3.3 Given a Markov transition semigroup P_t, acting on $\mathcal{B}_b(E)$, Dynkin introduces in [14] the space $\mathcal{B}_b^0(E) = \{f \in \mathcal{B}_b(E)$ such that $\lim_{t \to 0^+} P_t f(x) = f(x), \ x \in E\}$. Moreover he defines the *weak generator* $\tilde{L}$ of P_t as in (3.2), replacing $\mathcal{B}^0(E)$ with $\mathcal{B}_b^0(E)$. It is clear that $\tilde{L}$ extends the operator L given in (3.2). However, it seems a difficult problem to clarify if $\mathcal{B}_b^0(E) = \mathcal{B}^0(E)$ holds in general. Moreover, it is not clear how to prove an analogous of Proposition 3.2 when L is replaced by $\tilde{L}$. ▪

Let us apply the previous theorem to the generator of a Gaussian Ornstein-Uhlenbeck process X_t, which solves the SDE:

$$dX_t = AX_t dt + dW_t, \quad x \in H. \tag{3.4}$$

Here W_t is a $Q-$Wiener process with values in H and Q is a trace class operator on H, see also (2.2). Moreover A generates a $\mathcal{C}_0$-semigroup e^{tA} on H.

Define $\hat{C} \subset C_b^2(H)$ as the space of all functions f such that $Df(x) \in D(A^*)$, for all $x \in H$, and the functions $A^* Df$ and $D^2 f$ are both uniformly continuous and bounded on H.

Combining [34, Theorem 5.1] and Theorem 3.1, we get

Proposition 3.4 *Let us consider the Ornstein-Uhlenbeck semigroup P_t associated to the process X_t in (3.4). Then for any $f \in \hat{C}$, one has:*

$$\mathcal{A}f(x) = \frac{1}{2}\mathrm{Tr}\,(QD^2 f(x)) + \langle A^* Df(x), x\rangle = 0, \; x \in H \iff f \text{ is a BHF for } P_t.$$

Proof By the Ito formula, in [34] it is showed that, for any $f \in \hat{C}$, $f \in D(L)$ if and only if $\mathcal{A}f$ is bounded. Moreover if $f \in \hat{C} \cap D(L)$, then $Lf = \mathcal{A}f$. Using this result and Theorem 3.1, we finish the proof. ■

4 The Liouville theorem

If $A : D(A) \subset H \to H$ is a closed operator on H, we denote by $\sigma(A)$ its spectrum and by A^* its adjoint operator. We collect our assumptions on the generalised Mehler semigroup S_t, see (2.3) and (2.4).

Hypothesis 4.1 *(i) there exists $B_0 \in \mathcal{L}(U, H)$, where U is another Hilbert space, such that the linear nonnegative bounded operators $Q_t : H \to H$,*

$$Q_t x = \int_0^t e^{sA} B_0 B_0^* e^{sA^*} x\, ds, \; x \in H, \quad \text{are trace class}, \quad t > 0; \tag{4.1}$$

(ii) $\mu_t = \nu_t * N(0, Q_t)$, where ν_t is a family of probability measures on H, such that

$$\hat{\nu}_t(h) = \exp\Big(- \int_0^t \psi_1(e^{sA^*} h) ds\Big), \;\; h \in H, \; t \geq 0, \tag{4.2}$$

with $\psi_1 : H \to \mathbb{C}$ being a continuous, negative definite function such that $\psi_1(0) = 0$.

Hypothesis 4.2 *There exists $T > 0$, such that $e^{tA}(H) \subset Q_t^{1/2}(H), \;\; t \geq T$.*

If S_t is in particular the Gaussian Ornstein-Uhlenbeck semigroup corresponding to (3.4), then Hypothesis 4.2 is implied by the strong Feller property of S_t. Recall that a Markov semigroup P_t, acting on $\mathcal{B}_b(H)$, is called *strong Feller* if

$$P_t(\mathcal{B}_b(H)) \subset \mathcal{C}_b(H), \;\; t > 0. \tag{4.3}$$

Hypothesis 4.3 *One has:*

$$\int_H (\log |y| \vee 0)\, M(dy) < \infty. \tag{4.4}$$

Remark that if H is finite dimensional, then the previous hypotheses reduce to the assumptions in [27, Theorem 3.1].

The aim of this section is to prove the following theorem.

Theorem 4.1 *Let S_t be a generalised Mehler semigroup on H. If Hypotheses 4.1 and 4.2 hold and moreover*

$$s(A) := \sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} \leq 0, \tag{4.5}$$

then all BHFs for S_t are constant.
If Hypotheses 4.1, 4.2 and 4.3 hold and further

$$\sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} > 0,$$

then there exists a nonconstant BHF h for S_t.

Remark 4.2 As we mentioned in Introduction, a natural class of generalised Mehler semigroups which satisfy Hypotheses 4.1 and 4.2 is the one associated to the SDE

$$dX_t = AX_t dt + BdW_t + CdZ_t, \;\; X_0 = x \in H, \; t \geq 0, \tag{4.6}$$

where A generates a $\mathcal{C}_0$-semigroup e^{tA} on H, B and $C \in \mathcal{L}(U, H)$. Here W_t and Z_t are U-valued, independent Q_0-Wiener and Lévy processes (the operator Q_0 is a symmetric nonnegative trace class operator on U). Without any loss of generality, we may assume that Z_t has no Gaussian component (i.e., the exponent ψ_0 of Z_t is given by (2.2) with $Q = 0$).

It is well known that there exists a unique mild solution to (4.6), see [9] and [11]. This is given by

$$X_t^x = Y_t^x + \eta_t, \tag{4.7}$$

where

$$Y_t^x = e^{tA}x + \int_0^t e^{(t-s)A} B dW_s, \quad \eta_t = \int_0^t e^{(t-s)A} C dZ_s.$$

The latter stochastic integral involving Z_t can be defined as a limit in probability of elementary processes. Moreover Y_t^x is a Gaussian Ornstein-Uhlenbeck process, compare with (3.4). Clearly, setting $B_0 = B\, Q_0^{1/2}$, the operators B_0 and A satisfy condition (i) in Hypothesis 4.1.

If μ_t denotes the law of X_t^0, then it is clear that the Markov semigroup S_t associated to X_t^x is given by

$$S_t f(x) = \int_H f(e^{tA}x + y) \mu_t(dy), \;\; t \geq 0, \, x \in H, \, f \in \mathcal{B}_b(H). \tag{4.8}$$

If ν_t is the law of η_t then we have $\mu_t = \nu_t * N(0, Q_t)$. Indeed

$$\hat{\mu}_t(h) = \exp\Big(- \int_0^t |B_0^* e^{sA^*} h|^2 ds \Big) \; \exp\Big(- \int_0^t \psi_0(C^* e^{sA^*} h) \, ds \Big)$$
$$= \widehat{N(0, Q_t)}(h) \, \hat{\nu}_t(h), \quad h \in H. \; ∎$$

Remark 4.3 An example of a generalised Mehler semigroup with exponent ψ which is *not* Sazonov continuous, is the one determined by the SDE

$$dY_t = AY_t dt + B_0 dW_t, \;\; Y_0 = x \in H, \; t \geq 0, \tag{4.9}$$

where $A : D(A) \subset H \to H$ generates a $\mathcal{C}_0$-semigroup e^{tA} on H, $B_0 \in \mathcal{L}(U, H)$ and the process W_t is a U-valued *cylindrical* Wiener process, see [11] for more details.

If we assume that A and B_0 verify (i) in Hypothesis 4.1, then there exists a unique H-valued process Y_t^x, which is the mild solution to (4.9),

$$Y_t^x = e^{tA}x + \int_0^t e^{(t-s)A} B_0 dW_s, \;\; x \in H, \; t \geq 0. \tag{4.10}$$

Note that Y_t^x is a Gaussian process. The associated Ornstein-Uhlenbeck semigroup U_t is given by

$$U_t f(x) = \mathbb{E} f(Y_t^x) \; = \; \int_H f(e^{tA}x \, + \, y) \, \kappa_t(dy), \; f \in \mathcal{B}_b(H), \tag{4.11}$$

$x \in H, \; t > 0$, where $\kappa_t = N(0, Q_t)$ is the Gaussian measure on H with mean 0 and covariance operator Q_t, see (4.1). Note that the exponent ψ of U_t, i.e.,

$$\psi(y) = |B_0^* y|^2, \;\; y \in H,$$

is not Sazonov continuous unless the operator B_0 is Hilbert-Schmidt. However the associated process Y_t^x takes values in H, i.e., the function: $y \mapsto \int_0^t \psi(e^{sA^*}y)ds$ is Sazonov continuous on H, for each $t \geq 0$. ∎

Remark 4.4 One can show that the existence of an ergodic invariant probability measure with full support for a strong Feller transition semigroup implies the Liouville property. However, we are especially interested in cases in which there are no invariant probability measures. In particular if some $\lambda \in \sigma(A)$ is purely imaginary, then there are no invariant probability measures for the Ornstein-Uhlenbeck semigroup U_t given in (4.11), see [11], but still, under Hypothesis 4.2, the Liouville theorem holds. ∎

In the proof of the first statement of Theorem 4.1, we will need assertion (1) of the next result. This lemma also extends previous results proved in [11, Section 9.4] and in [28]. Recall that I_B denotes the indicator function of a set $B \subset H$.

Lemma 4.5 *Let us assume that Hypotheses 4.1 and 4.2 hold. Then one has:*
(1) $S_t(\mathcal{B}_b(H)) \subset C_b^\infty(H)$, *$t \geq T$.*
(2) S_t *is irreducible, i.e.,* $S_t I_O(x) > 0$, *for any $x \in H$, $t \geq T$ and O open set in H.*

Proof Take any $f \in \mathcal{B}_b(H)$. We have:

$$\begin{aligned} S_t f(x) &= \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) N(0, Q_t)(dy) \\ &= \int_H \nu_t(dz) \int_H f(y+z) N(e^{tA}x, Q_t)(dy), \; t \geq 0, \; x \in H. \end{aligned} \tag{4.12}$$

Using the Cameron-Martin formula, see [11], we can differentiate $S_t f$ in each direction $h \in H$ and get, for any $x \in H$, $t \geq T$,

$$\langle DS_t f(x), h\rangle = \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z)\langle Q_t{}^{-1/2}y, Q_t{}^{-1/2}e^{tA}h\rangle N(0, Q_t)dy. \quad (4.13)$$

Recall that the function: $y \mapsto \langle Q_t{}^{-1/2}y, Q_t{}^{-1/2}e^{tA}h\rangle$ is a Gaussian random variable on the probability space $(H, \mathcal{B}(H), N(0, Q_t))$, for any $t \geq T$, see [11] and [34].

Formulas similar to (4.13) can be easily established for higher order derivatives of $S_t f$. It is then straightforward to verify that $S_t f \in \mathcal{C}_b^\infty(H)$, $t \geq T$. This concludes the proof of the first statement.

The second statement follows since the measure $N(0, Q_t)$ has support on the whole H, for any $t \geq T$. ■

Proof of Theorem 4.1. *The first part.* Here we prove that any bounded harmonic function for S_t is constant.

By Hypothesis 4.2, the closed operators $Q_t^{-1/2}e^{tA}$ are bounded operators on H, for any $t \geq T$. They have also a control theoretic meaning, see for instance [33] or [10]. Note that (i) in Hypothesis 4.1 and Hypothesis 4.2 imply that the semigroup e^{tA} is compact, for any $t \geq T$. To see this, we write $e^{TA} = Q_T^{1/2}(Q_T^{-1/2}e^{TA})$ and remark that the operator $Q_T^{1/2}$ is Hilbert-Schmidt.

Thus we can apply the following result, which is proved in [25],

$$\lim_{t\to\infty} Q_t^{-1/2}e^{tA}x = 0, \;\; x \in H, \;\; \textit{if and only if} \;\; s(A) = \sup\{Re(\lambda) \;:\; \lambda \in \sigma(A)\} \leq 0. \quad (4.14)$$

Take any BHF f for S_t. We show that f is constant. By (4.13), we get the estimate:

$$\|\langle Df(\cdot), h\rangle\|_0 = \|\langle DS_t f(\cdot), h\rangle\|_0$$

$$\leq \|f\|_0 \int_H \nu_t(dz) \int_H |\langle Q_t{}^{-1/2}y, Q_t{}^{-1/2}e^{tA}h\rangle| N(0, Q_t)dy \;\leq\; |Q_t{}^{-1/2}e^{tA}h| \, \|f\|_0,$$

$t \geq T$, $h \in H$. Now letting $t \to \infty$ in the last formula, we get that f is constant, using (4.14). The assertion is proved.

The second part. Here we assume that $s(A) > 0$ and construct a nonconstant BHF h for S_t. It was already noted that Hypotheses 4.1 and 4.2 imply that e^{tA} is compact, for any $t \geq T$. Hence, see [15], pages 330 and 247, the spectrum $\sigma(A)$ consists entirely of eigenvalues of finite algebraic multiplicity, is discrete and at most countable. Moreover, for any $r \in \mathbb{R}$, the set

$$\{\mu \in \sigma(A) \;:\; \mathrm{Re}(\mu) \geq r\} \;\text{ is finite.} \quad (4.15)$$

It follows that there exists an isolated eigenvalue μ such that $s(A) = \mathrm{Re}(\mu)$. Using this fact, the claim follows by the next result. ■

Proposition 4.6 *Let S_t be a generalised Mehler semigroup on H. Assume that there exists an isolated eigenvalue μ of A with finite algebraic multiplicity and such that $Re(\mu) > 0$. Then there exists a nonconstant BHF h for S_t.*

Proof Let D_0 be the finite dimensional subspace of H consisting of all generalised eigenvectors of A associated to μ.

Let $P_0 : H \to D_0$ be the linear Riesz projection onto D_0 (not orthogonal in general),

$$P_0 x = \frac{1}{2\pi i} \int_\gamma (w - A)^{-1} x \, dw, \quad x \in H, \tag{4.16}$$

where γ is a circle enclosing μ in its interior and $\sigma(A)/\{\mu\}$ in its exterior, see for instance Lemma 2.5.7 in [10] and [15, page 245]. We have $H = D_0 \oplus D_1$, where $D_1 = (I - P_0)H$. The closed subspaces D_0 and D_1 are both invariant for e^{tA} and moreover $D_0 \subset D(A)$. We set $A_0 = AP_0$ and further $A_1 = A(I - P_0)$, where

$$A_0 : D_0 \to D_0, \quad A_1 : (D(A) \cap D_1) \subset D_1 \to D_1. \tag{4.17}$$

The operator A_0 generates a group e^{tA_0} on D_0 and A_1 generates a $\mathcal{C}_0$-semigroup e^{tA_1} on D_1. The projection P_0 commutes with e^{tA} and the restrictions of e^{tA} to D_0 and D_1 coincide with e^{tA_0} and e^{tA_1} respectively. Moreover on D_0 one has: $\sigma(A_0) = \{\mu\}$. By means of P_0, let us define a generalised Mehler semigroup S_t^0 on D_0,

$$S_t^0 f(a) = \int_H f(e^{tA} P_0 a + P_0 y) \mu_t(dy) = \int_{D_0} f(e^{tA_0} a + z)(P_0 \circ \mu_t)(dz),$$

where $t \geq 0$, $a \in D_0$, $f \in \mathcal{B}_b(D_0)$ and $(P_0 \circ \mu_t)$ is the probability measure on D_0 image of μ_t under P_0. Suppose that we find $g : D_0 \to \mathbb{R}$, such that

$$S_t^0 g(a) = g(a), \ \ a \in D_0, \tag{4.18}$$

i.e., g is a BHF for S_t^0. Then, defining $h(x) = g(P_0 x)$, $x \in H$, we get that h is a nonconstant BHF for S_t. Thus our aim is to construct a nonconstant BHF g for S_t^0. Note that

$$(P_0 \hat{\circ} \mu_t)(y) = \hat{\mu}_t(P_0^* y) = \exp\Big(- \int_0^t \psi(P_0^* e^{rA^*} y) dr \Big), \ \ y \in D_0.$$

Since D_0 is finite dimensional, the negative function $\psi_0 : D_0 \to \mathbb{C}$, $\psi_0(s) = \psi(P_0^* s)$, $s \in D_0$, corresponds to a Lévy process L_t with values in D_0 and defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The law ν_t of L_t verifies:

$$\hat{\nu}_t(y) = \exp(-t\psi(P_0^* y)), \ \ y \in D_0, \ t \geq 0.$$

Let us consider the process $\tilde{X}_t^a$ on D_0,

$$\tilde{X}_t^a = e^{tA_0} a + \int_0^t e^{(t-s)A_0} dL_t, \ \ t \geq 0, \ a \in D_0. \tag{4.19}$$

It is clear that the law of $\tilde{X}_t^0$ is just $(P_0 \circ \mu_t)$, $t \geq 0$. This implies that the Markov semigroup associated to $\tilde{X}_t^a$ is S_t^0.

We have reduced our initial problem of finding a nonconstant BHF for S_t to a corresponding finite dimensional problem. Now in order to construct a nonconstant function g such that (4.18) holds, we can apply [27, Proposition 3.6]. The proof is complete. ∎

Remark 4.7 Here we show a possible improvement of Hypothesis 4.3.

Let $\mathcal{F}(H)$ be the subspace of $\mathcal{L}(H)$ consisting of all finite rank operators R which commute with e^{tA}, i.e., $Re^{tA} = e^{tA}R, \quad t \geq 0$.

For any $R \in \mathcal{F}(H)$, M^R denotes the spectral Lévy measure on $\mathrm{Im}R = R(H)$ corresponding to ψ^R through formula (2.2), where $\psi^R(s) = \psi(R^*s)$, $s \in R(H)$; note that $\psi^R : R(H) \to \mathbb{C}$ is a continuous, negative definite function such that $\psi^R(0) = 0$. Moreover the image of μ_t under R, has characteristic function

$$(R \hat{\circ} \mu_t)(h) = \exp\Big(- \int_0^t \psi^R(e^{sA^*}h)ds\Big), \quad h \in R(H), \ t \geq 0.$$

It is straightforward to check that the second part of Theorem 4.1 continues to hold if Hypothesis 4.3 is replaced by the following weaker assumption:

$$\int_{P(H)} (\log|y| \vee 0)\, M^P(dy) < \infty, \quad \text{for any projection} \ \ P \in \mathcal{F}(H) \ \blacksquare$$

Remark 4.8 One can extend the definition of generalised Mehler semigroup and show that Theorem 4.1 holds true in this more general setting.

A *shifted generalised Mehler semigroup* P_t, acting on $\mathcal{B}_b(H)$, is given by

$$P_t f(x) = \int_H f(e^{tA}x + e^{tA}h - h + y)\mu_t(dy), \quad t \geq 0, \, x \in H, \, f \in \mathcal{B}_b(H), \tag{4.20}$$

compare with (2.3), where e^{tA} is a $\mathcal{C}_0$-semigroup on H, μ_t, $t \geq 0$, is a family of probability measures on H satisfying (2.4) and h is a fixed vector in H. It is straightforward to verify that P_t is a Markov semigroup acting on $\mathcal{B}_b(H)$.

An example of shifted generalised Mehler semigroup is the Markov semigroup P_t associated to the Markov process J_t^x,

$$J_t^x = X_t^{x+h} - h, \ \ t \geq 0, \, x \in H,$$

where X_t^x is the mild solution to (4.6). If in addition we assume that $h \in D(A)$, then J_t^x solves

$$dJ_t = AJ_t dt + Ahdt + BdW_t + CdZ_t, \ J_0 = x \in H, \ t \geq 0,$$

under the same assumptions of Remark 4.2.

There is a one to one correspondence between BHFs for S_t given in (2.3) and BHFs for P_t. Indeed if g is a BHF for P_t, then the function f, $f(y) = g(y-h)$, $y \in H$, is a BHF for S_t. Viceversa, if u is a BHF for S_t, then the function w, $w(z) = u(z+h)$, $z \in H$, is a BHF for P_t. This shows that Theorem 4.1, with the same assumptions on e^{tA}, B and μ_t, holds more generally when the generalised Mehler semigroup S_t is replaced by the semigroup P_t, given in (4.20), without any additional hypothesis on $h \in H$. $\blacksquare$

5 Convexity of positive harmonic functions

In this section we prove that positive harmonic functions for generalized Mehler semigroups are convex under suitable assumptions. This result can be regarded as a stronger version of the first part of Theorem 4.1, see in particular Corollary 5.3.

Theorem 5.1 *Assume Hypotheses 4.1 and 4.2 and consider the generalised Mehler semigroup S_t given in (2.3). Moreover suppose that*

$$s(A) = \sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} \leq 0. \tag{5.1}$$

holds. Then any positive harmonic function g for S_t is convex on H.

The following lemma is an extension of a result due to S. Kwapien [19] (proved by him in the Gaussian case with a similar proof).

Lemma 5.2 *Under Hypotheses 4.1 and 4.2, for any nonnegative function $f : H \to \mathbb{R}$, there results:*

$$S_t f(x+a) + S_t f(x-a) \geq 2C_t(a)\, S_t f(x), \quad x,\, a \in H, \tag{5.2}$$

where $C_t(a) = \exp\Big(-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big)$, $t > 0$.

Proof Using the notation in (4.12), we have:

$$S_t f(x) = \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) N(0, Q_t)(dy), \quad t \geq 0.$$

By the Cameron-Martin formula, one finds:

$$S_t f(x+a) = \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) \frac{dN_{e^{tA}a, Q_t}}{dN_{0,Q_t}}(y) N_{0,Q_t}(dy)$$

$$= \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2 + \langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y\rangle\Big] N_{0,Q_t}(dy).$$

It follows that

$$\begin{aligned}
&\frac{1}{2}(S_t f(x+a) + S_t f(x-a)) \\
&= e^{-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2} \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) \frac{1}{2}\Big(e^{\langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y\rangle} \\
&\qquad + e^{-\langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y\rangle}\Big) N_{0,Q_t}(dy) \\
&\geq \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big] \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) N_{0,Q_t}(dy) \\
&= C_t(a)\, S_t f(x). \quad \blacksquare
\end{aligned}$$

Proof of Theorem 5.1. By the previous lemma, we have:

$$\begin{aligned}
\frac{1}{2}(g(x+a) + g(x-a)) &= \frac{1}{2}(S_t g(x+a) + S_t g(x-a)) \\
\geq \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big] S_t g(x) &= \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big] g(x).
\end{aligned}$$

Passing to the limit as $t \to \infty$, we infer, see (4.14),

$$\frac{1}{2}(g(x+a) + g(x-a)) \geq g(x), \quad x,\, a \in H. \tag{5.3}$$

By a classical result due to Sierpinski, see [30], this condition together with the measurability of g imply the convexity of g. $\blacksquare$

Corollary 5.3 *Under the assumptions of Theorem 5.1, any bounded harmonic function g for S_t is constant on H.*

Proof We may assume that that $1-g$ is a nonnegative BHF (otherwise replace g by $\frac{g}{\|g\|_0}$). Using $1-g$ instead of g in (5.3), we obtain:

$$\frac{1}{2}(1-g(x+a)+1-g(x-a)) = 1-\frac{1}{2}(g(x+a)+g(x-a)) \geq 1-g(x)$$

It follows that $g(x+a)+g(x-a) \leq 2g(x)$ and so, by (5.3),

$$g(x+a)+g(x-a) = 2g(x), \quad x \in H. \tag{5.4}$$

Note that, by Lemma 4.5, g is continuous on H. Since any continuous function which satisfies identity (5.4) is affine, we have $g(x) = g(0) + \langle h, x\rangle$ for some $h \in H$. It follows that g is constant. ■

6 Open questions

Problem 1. It is not known, even in finite dimension and for strong Feller Gaussian Ornstein-Uhlenbeck semigroups P_t, if the hypothesis

$$\sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} \leq 0$$

implies that all PHFs for P_t are constant (compare with Theorems 4.1 and 5.1).

A partial positive answer can be given in $\mathbb{R}^2$, see [8], and more generally in $\mathbb{R}^n$, assuming in addition that the dimension of the Jordan part of A corresponding to eigenvalues in the imaginary axis is at most two. This condition is equivalent to the recurrence of a strong Feller Gaussian Ornstein-Uhlenbeck process X_t in $\mathbb{R}^n$, see [13], [16] and [33]. Remark that for recurrent processes with strong Feller transition semigroups all positive harmonic functions, or even more generally all excessive functions, are constant, see [4].

We also mention the following related result, which has been recently proved in [18]. Let L be the Ornstein-Uhlenbeck operator on $\mathbb{R}^n$,

$$Lu(x) = \frac{1}{2}\text{Tr }(QD^2u(x)) + \langle Ax, Du(x)\rangle, \quad x \in \mathbb{R}^n,$$

where Q and A are real $n\times n$ matrices and Q is symmetric and nonnegative definite. Assume that L is hypoelliptic (or equivalently that the corresponding Ornstein-Uhlenbeck semigroup P_t is strong Feller, see for instance [20]). In [18] it is shown that if 0 is the only eigenvalue of A and if in addition the matrix Q is degenerate, then any nonnegative classical solution to $Lu(x) = 0$, $x \in \mathbb{R}^n$, is constant on $\mathbb{R}^n$.

Problem 2. Given a generalised Mehler semigroup S_t, acting on $\mathcal{B}_b(H)$, it is an open problem to find conditions on the drift operator A and on the exponent ψ in order to construct a càdlàg Markov process Y_t with values in H, having S_t as the associated Markov semigroup. In [17] such a process is constructed only on an enlarged Hilbert space E, containing H.

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